Bayesian Statistics
Part I: History, Philosophy, and Motivation
Part II: Introduction to Probability

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March 23, 2021

Who was Thomas Bayes?

 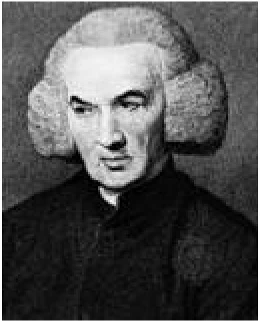 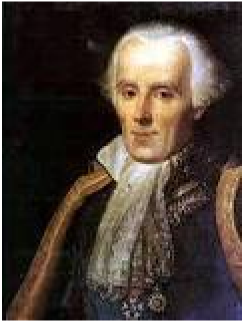

Figure: Major players in creation of Bayes' rule

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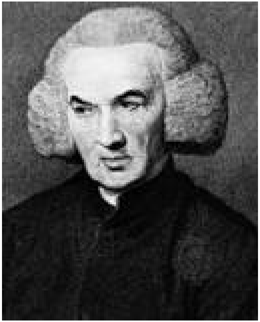

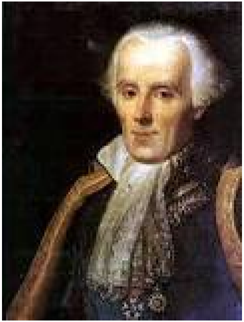

Figure: Major players in creation of Bayes' rule

Sharon Bertsch McGrayne †

- **McGrayne Bios (5:41-9:55)**

† "The Theory That Would Not Die" How Bayes' Rule Cracked the Enigma Code, Hunted Down Russian Submarines, and Emerged Triumphant from Two Centuries of Controversy"

Bayesian Paradigm

Initial Belief (prior)

Bayesian Paradigm

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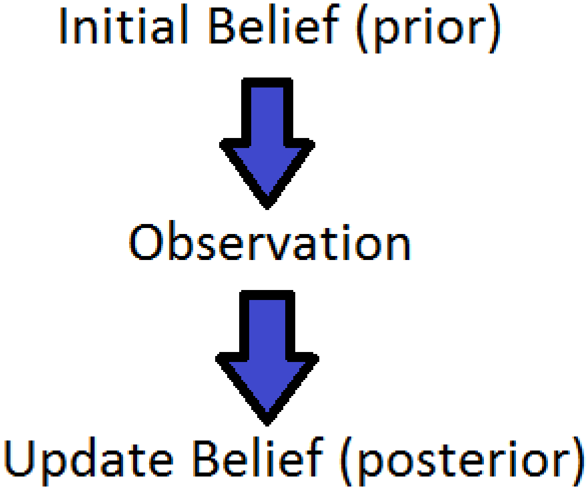

Motivation and History

The allegory of our statistical lives

- Most of us are born Bayesians.

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- **McGrayne WWII (16:02-30:23)**
- **McGrayne Air France Flight 447 (2:30-4:00)**

Motivation and History

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- Today's message- "It's never too late to have a happy childhood."

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Gender	Gender
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0	0
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0	0
1	0
1	0
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0	0
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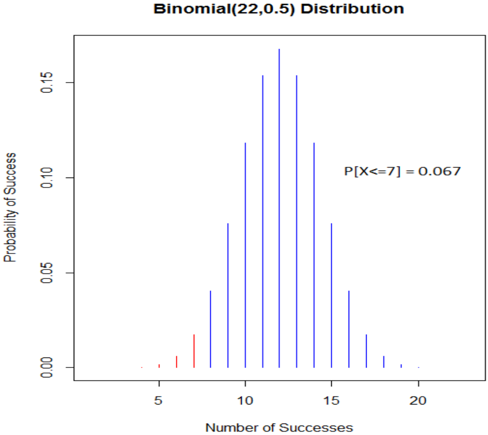

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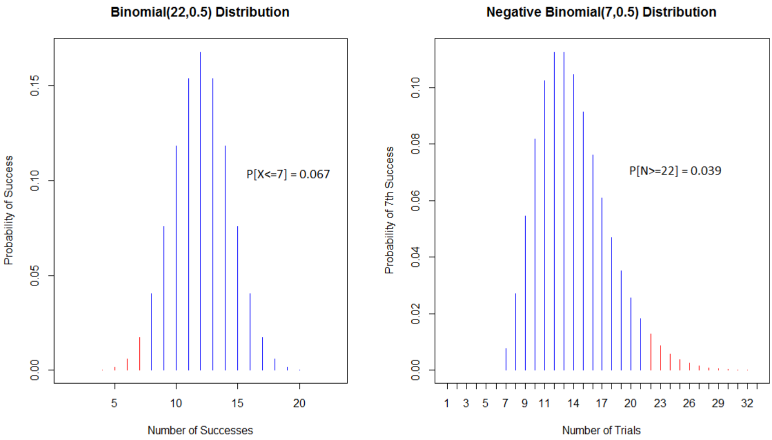

Figure: P-values depend on the researcher's intention.

General Bayes' Theorem

Bayes' Theorem

$$p(\theta|x_1,\ldots,x_n) = \frac{p(\theta)p(x_1,\ldots,x_n|\theta)}{\int p(\theta)p(x_1,\ldots,x_n|\theta)d\theta} \tag{1}$$

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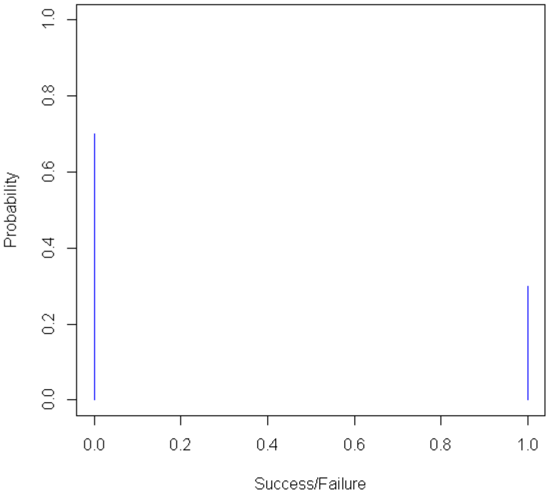
- Example: Plot a Bern(0.3)
  - `x<-c(0,1)`
  - `plot(x,dbinom(x,1,0.3),ylim=range(0,1),type="h", ylab="Probability",xlab="Success/Failure",col="blue")`

Bernoulli Distribution



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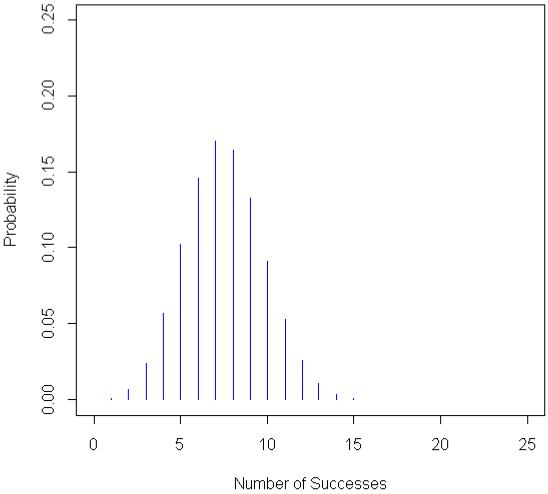
- Example: Plot a Bin(25,.3)
  - `x<-0:25`
  - `plot(x,dbinom(x,25,.3),ylim=range(0,.25),type="h", ylab="Probability",xlab="Number of Successes",col="blue")`

Binomial Distribution



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- Example: Plot a Uniform(0,1)
  - `x<-seq(0,1,length=1000)`
  - `plot(x,dunif(x,0,1),ylim=range(0,1),type="l", ylab="Density",xlab="Possible Outcome Values",col="blue")`

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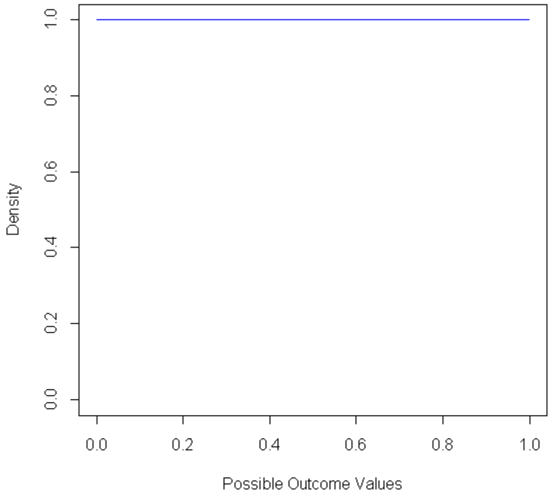

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- Example: Plot a N(0,1)
  - `x<-seq(-4,4,length=1000)`
  - `plot(x,dnorm(x,0,1),ylim=range(0,.25),type="l", ylab="Density",xlab="Possible Outcome Values",col="blue")`

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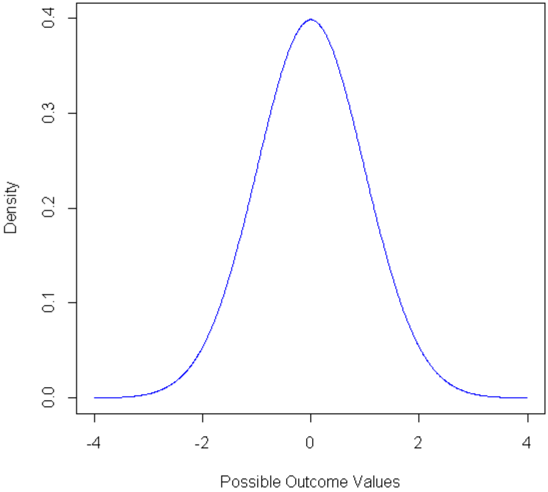

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- Example: Plot a Beta(3,5)
  - `x<-seq(0,1,length=1000)`
  - `plot(x,dbeta(x,3,5),ylim=range(0,3),type="l", ylab="Density",xlab="Possible Outcome Values",col="blue")`

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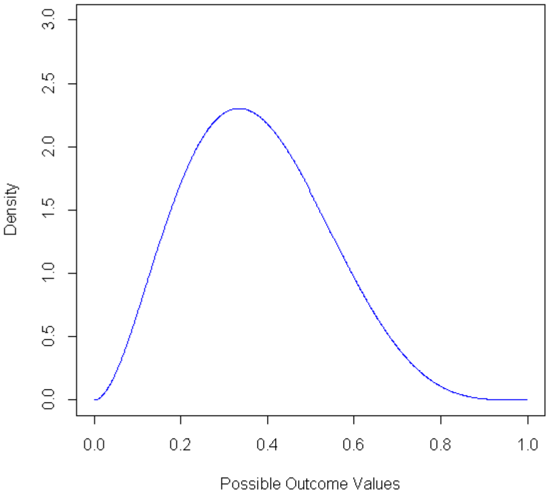

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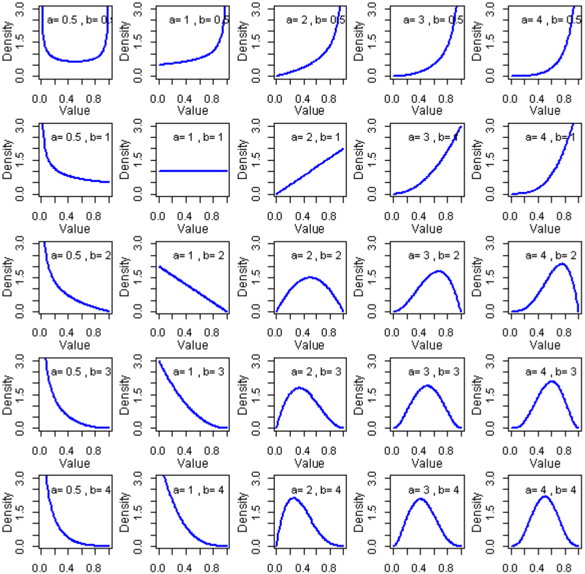

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- Recall that a Bernoulli pmf is defined by $p(x|\theta) = \theta^x(1-\theta)^{1-x}$.
- Then in a sample made up of n independent Bernoulli trials, the joint distribution is given by

$$\prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{1-x_i} = \theta^{\sum x_i}(1-\theta)^{n-\sum x_i}$$

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The maximum of the likelihood function is seen as the most plausible value of θ, given the data that was observed.

Estimating a Single Proportion: Surgery Example

Data is collected from various hospitals in the UK that perform cardiac surgery on babies between 1991 and 1995. The surgery center in Bristol reports 41 deaths and 143 operations during this time. Using vague priors for the proportion of deaths at this surgery center, estimated the posterior prevalence of mortality. See the files

- surgery.data.txt
- surgery.bugs.txt
- surgery.R